Journal of Statistical Physics, Vol. 26, No. 3, 1981

# Random Walk in an Inhomogeneous Medium with Local Impurities

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Received March 26, 1981

In spite of Sinai's result that the decay of the velocity autocorrelation function for a random walk on $\mathbb{Z}^d$ ($d = 2$) can drastically change if local impurities are present, it is shown that local impurities can not abolish weak convergence to the Brownian motion if $d \geqslant 2$.

**KEY WORDS:** Random walk; local impurities; random environment; weak convergence; Brownian motion.

## 1. INTRODUCTION

It has been observed recently that local impurities situated in a bounded domain can radically change the decay of the velocity autocorrelation function of a random walk.[1] For dimensions 1 and 2, this phenomenon can be explained heuristically by Pólya's theorem on the recurrence of the random walk, that can cause a long memory in the velocity autocorrelation function. The aim of the present paper is to prove that in dimension 2 local —possibly random—impurities do not influence the weak convergence of the random walk to the Brownian motion. The same is true in higher dimensions, too, while on the other hand a simple example is given showing that, on the line, the presence of as few as one impurity spoils the Brownian character of the weak limit.

If the impurities are not restricted to a finite domain, then the limit process, even if it happens to be a Brownian one, is expected to have a different covariance matrix. Convergence to the Brownian motion is only proven in a quite simple case: the impurities are deterministic and periodically situated.

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0022-4715/81/1100-0527$03.00/0 © 1981 Plenum Publishing Corporation

The results are formulated in Section 2, and they are proved in Section 4. Section 3 contains some notations and lemmas used in Section 4 while some of the later ones are proved in the Appendix. Section 5 is devoted to comments.

## 2. RESULTS

First of all we define the simple symmetric random walk on $\mathbb{Z}^d$, the $d$-dimensional integer lattice.

**Definition 1.** Let $y_1, y_2, \ldots$ be i.i.d. (independent, identically distributed) random variables

$$\text{Prob}(y_i = \pm e_j) = \frac{1}{2d}, \qquad i \in \mathbb{N}, \qquad j = 1, 2, \ldots, d$$

where the $e_j$ is the $j$th unit vector. The stochastic process $Y_0, Y_1, \ldots$ defined by

$$Y_n = y_0 + \sum_{i=1}^{n} y_i$$

where $y_0 = z \in \mathbb{Z}^d$, is called a *simple symmetric random walk* (SSRW). The measure defined by this random walk will be denoted by $\underline{P}_z$ and the transition probabilities of the process will be denoted by $\underline{P}(x, y)$.

It is well known that the continuous time stochastic process $\eta_n(t) = n^{-1/2} Y_{[nt]}$, $t \in [0, 1]$ converges, as $n \to \infty$, weakly to $W(t)$, the $d$-dimensional standard Wiener process, in the space $C^d[0, \infty)$. Weak convergence in $C^d[0, \infty)$ will be denoted by $\Rightarrow$.

**Definition 2.** Let $P$ be a transition probability matrix on $\mathbb{Z}^d$ such that $P(x, y) = 0$ if $|x - y| \neq 1$. We call the Markov process $X_i$, $i = 0, 1, \ldots$ with transition probabilities $P$ a *simple random walk in an inhomogeneous medium*.

**Definition 3.** If $X_n$ is a simple random walk in an inhomogeneous medium and there exists a finite set $A \subset \mathbb{Z}^d$ such that for all $u \notin A$, $v \in \mathbb{Z}^d$, $P(u, v) = \underline{P}(u, v)$, then we call $X_n$ a *simple, symmetric random walk with local impurities*. (Abbreviated as R.W.w.L.I.)

**Theorem 1.** Let $X_n$ be a R.W.w.L.I., where the starting point $z$ lies in the infinite, strongly connected component $Q$ of the graph $G = (\mathbb{Z}^d, E)$, where $E = \{(u, v) \mid P(u, v) \neq 0\}$. If $d \geq 2$ and $U_n(t) = n^{-1/2} X_{[nt]}$, $t \in [0, 1]$ then, as $n \to \infty$,

$$U_n(t) \Rightarrow W(t), \qquad t \in [0, 1]$$

*Remark.* If $d = 1$, this theorem is not true. This is shown by the following example.

*Example.* Let $A = \{\mathbb{O}\}$ and $P(\mathbb{O}, 1) = 1 - P(\mathbb{O}, -1) = p$, $p \in (0, 1)$. Let $p(x)$ be the following function defined on $\mathbb{Z}$:

$$p(x) = \begin{cases} 2p & \text{if } x > 0 \\ 2(1-p) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The process $U_n(t)$, in our example, converges weakly to a process $\gamma(t)$, whose one-dimensional density is equal to $p(x) w_1(t, x)$, where $w_1(t, x)$ is the corresponding density of the Wiener process. This statement can be easily proved by using the following equations:

$$P_0(X_n = y, X_i \neq 0, 1 \leqslant i < n) = p(y) \underline{P}_0(Y_n = y, Y_i \neq 0, 1 \leqslant i < n)$$

$$P_0(X_n = 0) = \underline{P}_0(Y_n = 0)$$

and

$$P_0(X_n = y) = p(y) \underline{P}_0(Y_n = y)$$

The finite-dimensional densities of $\gamma(t)$ can also be calculated by using the well-known reflection principle.

If $X_n$ is a R.W.w.L.I. and the impurities are contained in a finite set $A$, then a similar statement is true. The only difference is that the limit process is determined by the probabilities

$$P(X_\xi = a_1 - 1 \mid X_0 = a_1), \qquad P(X_\xi = a_2 + 1 \mid X_0 = a_1)$$

$$P(X_\xi = a_1 - 1 \mid X_0 = a_2), \qquad P(X_\xi = a_2 + 1 \mid X_0 = a_2)$$

where $a_1 = \min\{z \in A\}$, $a_2 = \max\{z \in A\}$ and $\xi = \min\{k \in \mathbb{N} \mid X_k \notin A\}$ is the first exit time from the set $A$.

**Definition 4.** Let $Z_n$ be a finite Markov chain on a finite set $S$. Let $h : S \to \mathbb{R}^d$ be an arbitrary function. For simplicity we assume that $S$ is the unique essential class of the Markov chain. If $X_0 \in \mathbb{R}^d$ and $X_n = X_0 + \sum_{i=1}^n h(Z_i)$, then we call $X_n$ a *random walk directed by the Markov chain* $Z_n$.

**Proposition 2.** If $X_n$ is as in Definition 4, then there exists an $M \in \mathbb{R}^d$, a covariance matrix $\Sigma$ such that for the processes

$$V_n(t) = n^{-1/2}(X_{[nt]} - [nt] M), \qquad t \in [0, 1]$$

we have, as $n \to \infty$,

$$V_n(t) \Rightarrow W_\Sigma(t), \qquad t \in [0, 1]$$

where by $W_\Sigma(t)$ we denote the Wiener process with zero expectation and covariance matrix $\Sigma$.

**Definition 5.** Let $G$ be a subgroup of $\mathbb{Z}^d$ of finite index and denote by $A = \mathbb{Z}^d / G$ its fundamental parallelepiped. Suppose that $P$ is an $A$-

periodic transition probability matrix on $\mathbb{Z}^d$, i.e., $P(x, y) = P(u, v)$, whenever $x \equiv u \pmod A$ and $x - u = y - v$.

If $X_n$ is a simple random walk in an inhomogeneous medium and its transition probability matrix is $A$-periodic, then we call $X_n$ *a simple random walk in a periodic medium.*

It is clear that the mod$A$ factorization of $X_n$ generates a finite Markov chain $\zeta_n$ on $A$ and $X_n$ is directed by $\zeta_n$. So we get

**Corollary 3.** If $X_n$ is a simple random walk on $\mathbb{Z}^d$ $(d \geqslant 1)$ in a periodic medium, and $X_0 = z \in \mathbb{Z}^d$ is arbitrary, then there exists an $M \in \mathbb{R}^d$ and a covariance matrix $\Sigma$ such that, if $Z_n(t) = n^{-1/2}(X_{[nt]} - [nt]M)$ then, as $n \to \infty$,

$$Z_n(t) \Rightarrow W_\Sigma(t), \qquad t \in [0, 1]$$

## 3. PRELIMINARY NOTES TO THE PROOF OF THEOREM 1

Before the exact discussion we want to show the idea of the result. If $d \geqslant 3$, then Pólya's theorem[3] says that with probability 1 the number of returns into the origin and into a finite set $A$ as well is finite. Therefore in case of a finite modification the random walk leaves the set $A$ after a finite time. Thus, in the limit, the effect of the modification vanishes. In the case $d = 2$, the expected number of returns into a finite set until time $n$ is $o(\sqrt{n})$. The expected time spent in $A$ during one visit is bounded and since the normalizing factor is of order $\sqrt{n}$, the previous conclusion is also true.

If the impurities are local, then there exists an $N \in \mathbb{N}$ such that all the impurities lie in the cube

$$K_N = \left[-N - \tfrac{1}{2}; N + \tfrac{1}{2}\right]^d \subset \mathbb{R}^d, \qquad \text{i.e.,} \quad A \subset K_N$$

**Definition 6.**

$$\theta_n = \sum_{i=0}^{n} \delta(Y_i, O)$$

(where $\delta$ is the Kronecker symbol), i.e., $\theta_n$ is the number of visits to the origin in the first $n$ steps of the simple symmetric random walk.

**Definition 7.** For all $z \in K_N \cap Q$ consider a R.W.w.L.I. such that $X_0 = z$. Denote

$$\tau_z = \min\{k \in \mathbb{N} \mid X_k \notin K_N\}$$

i.e., $\tau_z$ is the first exit time of the R.W.w.L.I. from the set $K_N$.

**Definition 8.**

$$\rho_n = \sum_{i=0}^{n} \chi_{X_i \in K_N}$$

the time spent by the R.W.w.L.I. in $K_N$ until time $n$.

**Definition 9.** Denote by $\nu_n$ the number of pure 1 blocks in the sequence

$$\chi_{X_0 \in K_N}, \chi_{X_1 \in K_N}, \ldots, \chi_{X_n \in K_N}$$

i.e., $\nu_n$ is the number of entrances into $K_N$ until time $n$.

**Definition 10.** Denote by $B \subset Q$ a finite simply connected set, i.e., for all $x, y \in Q \setminus B$ there is a directed path from $x$ to $y$ in the subgraph on $Q \setminus B$. Let

$$S_B = \min\{k \in \mathbb{N} \mid X_k \in B\}$$

and

$$T_B = \min\{k \in \mathbb{N} \mid Y_k \in B\}$$

These are called hitting times of $B$ by the processes $X_n$ and $Y_n$, respectively.

**Lemma 4.** For all $z \in Q \setminus B$ the following limit exists:

$$0 < \lim \frac{P_z(T_B > n)}{\underline{P}_0(T_{\{0\}} > n)} < \infty$$

This is a variant of the Kesten–Spitzer ratio limit theorem.[3]

We shall denote by $\underline{E}_z$ (and $\underline{E}_z$) expectations with respect to $P_z$ (and $\underline{P}_z$). To prove Theorem 1 we need the following lemmas:

**Lemma 5.** If $d \geqslant 2$, then, as $n \to \infty$,

$$\underline{E}_z(\theta_n) = o(\sqrt{n})$$

Let, for $A \subset \mathbb{Z}^d$, $\partial A = \{x \in A \mid \exists y \in \mathbb{Z}^d \setminus A : |x - y| = 1\}$.

**Lemma 6.** If $d \geqslant 2$ and there exists a $b \in \partial(\mathbb{Z}^d \setminus K_N)$ such that for all $x \in K_N$, $y \in \mathbb{Z}^d \setminus K_N P(x, y) = 0$ if $y \neq b$, then for all $z \in \mathbb{Z}^d$

$$E_z(\nu_n) \leqslant \frac{1}{P_b(T_{K_N} > n)}$$

**Corollary 7.** If $d \geqslant 2$ then, as $n \to \infty$, $E_z(\nu_n) = o(\sqrt{n})$.

**Lemma 8.** If $d \geqslant 2$, $z \in Q$, then, as $n \to \infty$,

$$E_z(\rho_n) = o(\sqrt{n})$$

## 4. PROOF OF THE RESULTS

*Proof of Theorem 1.* With the help of $X_n$ we define a new process $U_n$. Let $U_0 = z$ and $U_{n+1} - U_n = X_{n+1} - X_n$ if $X_n \notin K_N$, while if $X_n \in K_N$, then let $U_{n+1} - U_n$ be independent of $X_0, X_1, \ldots, X_n, \ldots$ and of $U_0, U_1, \ldots, U_n$, and let $P(U_n, U_{n+1}) = \underline{P}(U_n, U_{n+1})$.

It is clear that $U_n$ is a simple symmetric random walk.

Since $X_n$ spends time $\rho_n$ in $K_N$ until time $n$

$$|X_n - X_0 - U_n| \leqslant 2\rho_n$$

From Lemma 8 we get that

$$E\Big(\sup_{0 \leqslant t \leqslant 1} \rho_{[nt]}\Big) = o(\sqrt{n})$$

and by the Markov inequality it follows that

$$n^{-1/2} \sup_{0 \leqslant t \leqslant 1} |X_{[nt]} - U_{[nt]}|$$

tends to zero in probability if $n \to \infty$.

Thus the weak convergence of the normalized SSRW to the Wiener process implies the statement of our theorem. ■

*Proof of Proposition 2.* It is enough to refer to Theorem 20.1 and Example 2 in Ref. 4. $X_n$ is a function of a finite Markov chain and consequently $X_n$ is exponentially mixing. By Theorem 20.1 and by the Cramer–Wold device we get the statement. ■

## 5. COMMENTS

It is easy to see that Theorem 1 also implies a generalization to random walks $X_n$ in uniformly local random media when a bounded set $A \subset \mathbb{Z}^d$ and a measurable $\sigma$ algebra $\mathfrak{F}$ is given such that (i) conditioned with respect to $\mathfrak{F}$ the random walk is a simple, symmetric random walk with local impurities; and (ii) the impurities lie in $A$ with probability 1. For such random walks the statement of Theorem 1 is true if—for the sake of simplicity—we suppose the $X_0 = z \in \mathbb{Z}^d \backslash A$. The velocity autocorrelation function of $X_n$ has been obtained by Ya. G. Sinai (private communication).

If the random impurities are not local, then the non-Markovian character of the process makes it hard to handle. There exist several, more or less related models and, to our knowledge, even simple questions—like the existence of a positive diffusion constant—are not answered for any of them, except for Sinai's recent result for a one-dimensional model.[5] We can call a discrete-time stochastic process on $\mathbb{Z}^d$ a simple random walk in a random medium if for a suitably chosen $\sigma$ algebra $\mathfrak{F}$ the process conditioned with respect to $\mathcal{F}$ is a simple random walk. In Sinai's model the random transition probabilities are determined by a constant $\epsilon$ $(0 < \epsilon < 1/2d)$ and by a Bernoulli sequence of random variables $\xi_{x,l}$ $(x \in \mathbb{Z}^d, 1 \leqslant l \leqslant d)$ taking on the values $-1$ or $+1$ in such a way that for every $x \in \mathbb{Z}^d$ and $1 \leqslant l \leqslant d$

$$P(x, x \pm e_l \mid \mathcal{F}) = 1/2d \pm \epsilon \xi_{x,l}$$

where $\mathcal{F}$ is the $\sigma$ algebra spanned by the $\xi_{x,l}$'s. Another model has been proposed by Spohn (see Ref. 2): cancel the points of $\mathbb{Z}^d$ independently of each other with probability $P$; if $P$ is small, then with positive probability there exists an infinite connected component $Q$ of the noncanceled points, and on this random component a simple random walk will be defined by saying that the particle chooses any of its $Q$-neighbors with equal probabilities. We do not know of any results concerning this model.

To treat recurrence properties of random walks in random media the present authors introduced the following model: to every edge $E = \{(i, j) \mid i, j \in \mathbb{Z}^d, |i - j| = 1\}$ of the unoriented lattice graph, we define independent, identically distributed random variables $\alpha_E$ and then

$$P(i, j \mid \mathcal{F}) = \alpha_{(i, j)} \Big/ \sum_{k \,:\, |k - i| = 1} \alpha_{(i,k)}$$

where $\mathcal{F}$ is generated by the $\alpha_E$'s.

We return to the discussion of our results in a forthcoming paper.

## ACKNOWLEDGMENT

The authors express their gratitude to L. A. Bunimovich for making many useful remarks.

## APPENDIX. (PROOF OF THE LEMMAS)

*Proof of Lemma 5.*

$$\text{If } d = 2, \text{ then } \underline{E}_0(\theta_n) \sim (1/\pi)\log n \tag{1}$$

(See Ref. 3, p. 200.)

$$\text{If } d \geqslant 3, \text{ then } \underline{E}_0(\theta_n) \sim C_d < \infty \tag{2}$$

We note that, if $d = 1$, then $\underline{E}_0(\theta_n) \sim (2/\pi)\sqrt{n}$. ■

*Proof of Lemma 6.* We define a complete system of events $A_i$, $i = 0, 1, \ldots, n$:

$$A_i = \{X_i = b \text{ and } X_j \notin K_N \text{ for } i < j \leqslant n\}$$

Then

$$1 = \sum_{i=0}^{n} P_z(A_i) = \sum_{i=0}^{n} P_z(X_i = b) P_b(S_{K_N} > n - i)$$

$$\geqslant P_b(S_{K_N} > n) \cdot E(\nu_n) \qquad \text{since } E(\nu_n) \leqslant \sum_{i=0}^{n} P_z(X_i = b)$$

and we get that

$$E_z(\nu_n) \leqslant \frac{1}{P_b(S_{K_N} > n)}$$

Let us notice that the probability $P_b(S_{K_N} > n)$ is independent of the transition probabilities inside $K_N$; consequently

$$\underline{P}_b(T_{K_N} > n) = P_b(S_{K_N} > n)$$

which is the statement. ■

*Proof of Corollary 7.* It is known that, if $d = 2$, (see Ref. 3, p. 199),

$$P_0(T_{\{O\}} > n) \sim \frac{\pi}{\log n} \tag{3}$$

and, if $d \geqslant 3$, then

$$\lim_{n\to\infty} P_b(T_{K_N} > n) > 0 \tag{4}$$

From (1)–(4) and Lemmas 4 and 6 it follows that

$$\lim_{n\to\infty} \frac{E_z(\nu_n)}{\underline{E}_0(\theta_n)} \leqslant \lim_{n\to\infty} \frac{\underline{P}_0(T_{\{0\}} > n)}{\underline{P}_b(T_{K_N} > n)} < \infty \quad ■$$

It is clear that Lemma 6 and its Corollary 7 remain true if we drop the condition that for $x \in K_N, y \in \mathbb{Z}^d P(x, y) = 0$ if $|x - y| \neq 1$. We will use this fact in the proof of Lemma 8.

*Proof of Lemma 8.* Let $\eta_{i-1}$ be the time spent by $X_n$ outside $K_N$ between the $(i-1)$th and the $i$th visit to $K_N$. It is the length of the $i$th 0-block in the sequence $\chi_{X_0 \in K_N}, \chi_{X_1 \in K_N}, \ldots$ . Let $\xi_{i-1}$ be the time spent by $X_n$ in $K_N$ i.e., the length of the $i$th 1-block.

Let $\eta_0 = 0$ if $X_0 \in K_N$. From Definition 9,

$$\nu_n = \min\left\{ k \in \mathbb{N} \mid \sum_{i=0}^{n} \xi_i + \eta_i > n \right\}$$

$\nu_n$ is, of course, a Markov stopping time with respect to

$$\mathcal{F}_i = \sigma(\xi_0, \xi_1, \ldots, \xi_i, \eta_0, \eta_1, \ldots, \eta_i)$$

We shall construct a new R.W.w.L.I. $Z_n$. Later for $Z_n$ we shall also define certain random variables $\bar{\xi}_n, \bar{\eta}_n, n = 0, 1, \ldots,$ and the following random variables will make sense:

$$\bar{\nu}_n = \min\left\{ k \in \mathbb{N} \mid \sum_{i=0}^{k} \bar{\xi}_i + \bar{\eta}_i > n \right\}$$

and

$$\bar{\rho}_n = \sum_{i=0}^{\bar{\nu}_n} \bar{\xi}_i$$

$\bar{\xi}_i$ and $\bar{\eta}_i$ will be defined to satisfy the inequalities $\xi_i \leqslant \bar{\xi}_i$ and $\eta_i \geqslant \bar{\eta}_i$ for $i = 0, 1, \ldots$ and as a consequence we obtain

$$\rho_n \leqslant \bar{\rho}_n$$

The random variables $\bar{\xi}_i$ and $\bar{\eta}_i$, $i = 0, 1, \ldots,$ will be totally independent with the $\bar{\xi}_i$'s (and the $\bar{\eta}_i$'s), respectively, identically distributed. $\bar{\nu}_n$ will be a Markov stopping time with respect to the sequence of $\sigma$ algebras $\sigma(\bar{\xi}_0, \bar{\xi}_1, \ldots, \bar{\xi}_i, \bar{\eta}_0, \bar{\eta}_1, \ldots, \bar{\eta}_i) = \bar{\mathcal{F}}_i$ and from the construction it will be clear that $E(\bar{\xi}_1) < \infty$.

Then by the Wald identity

$$E_z(\rho_n) < E(\bar{\rho}_n) = E(\bar{\xi}_1)E(\bar{\nu}_n). \qquad (*)$$

Finally, if we apply Corollary 7 to the process $Z_n$, we have

$$E(\bar{\nu}_n) = o(\sqrt{n}) \qquad \text{as } n \to \infty$$

and thus the previous inequality gives the statement of the lemma.

Let us define $Z_n$.

**Definition: $b_i$.** Let us denote the elements of $\partial K_N$ by $b_1, b_2, \ldots, b_q$, where $q = |\partial K_N|$.

**Definition: $M$.** $M > 2N, M \in \mathbb{N}$.

**Definition: $K^{(i)}$.** We take $q$ disjoint, identical copies of the set $K_N$. These sets will be denoted by $K^{(1)}, K^{(2)}, \ldots, K^{(q)}$.

**Definition: $b^{(i)}$.** In each $K^{(i)}$ there is a point corresponding to $b_i$ which will be denoted by $b^{(i)}$. Thus $b^{(i)} \in \partial K^{(i)}$, the subset of $K^{(i)}$ corresponding to $\partial K_N$.

**Definition: $X^{(i)}$.** If $X \in K_N$ then for every $i = 1, \ldots, q$ we shall denote by $X^{(i)}$ the element of $K^{(i)}$ corresponding to $X$.

Next we define a random walk on the state space

$$I = \bigcup_{i=1}^{q} K^{(i)} \cup (\mathbb{Z}^d \setminus K_M)$$

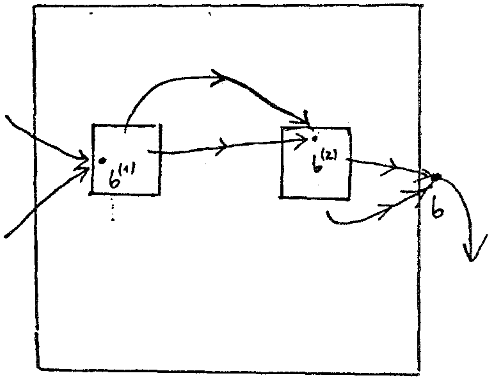


Fig. 1.

**Definition: $\bar{P}$.**

(1) $\bar{P}(x, y) = \underline{P}(x, y)$ if $x, y \in \mathbb{Z}^d \setminus K_M$;

(2) $\bar{P}(x, y) = P(u, v)$ if for some $i \in \{1, \ldots, q\}$ $x = u^{(i)}$, $y = v^{(i)}$;

(3) $\bar{P}(x, b^{(1)}) = \dfrac{1}{2d}$ if $x \in \partial(\mathbb{Z}^d \setminus K_M)$;

For $i = 1, 2, \ldots, q-1$

(4i) $\bar{p}(x, b^{(i+1)}) = \sum_{\substack{u, v \,:\, v \notin K_N \\ |u-v|=1}} P(u, v)$ if $x = u^{(i)}$

with $u \in \partial K_N$;

Let $b = (M+1)e_1 \in \partial(\mathbb{Z}^d \setminus K_M)$ be the center of a side of $K_{M+1/2}$.

(5) $\bar{P}(x, b) = \sum_{\substack{u, v \,:\, v \notin K_N \\ |u-v|=1}} P(u, v)$ if $x \in \partial K^{(q)}$ and $u^{(q)} = x$.

(6) All the remaining transition probabilities are equal to zero. We illustrate the graph of $\bar{P}$ in Fig. 1 for $q = 2$. The arrows show the possible steps between $K^{(i)}$ and $K^{(i+1)}$ or $K^{(q)}$ and $\mathbb{Z}^d \setminus K_M$.

**Definition: $Z_n$.** Let $Z_0 = b$ and let us consider the R.W.w.L.I. to be generated by $\bar{P}$.

**Definition: $\alpha_j^{(i)}$, $\beta_j$, $\bar{\xi}_j$, $\bar{\eta}_j$.** Let $\alpha_j^{(i)}$ $(j \geqslant 0, 1 \leqslant i \leqslant q)$ be the time of the $j$th visit of $Z_n$ at the point $b^{(i)}$ $(\alpha_0^{(1)} = 0)$ and $\beta_j$ be the time of the $j$th visit of $Z_n$ at the point $b$. Clearly $\alpha_j^{(i)} < \alpha_j^{(i+1)} < \beta_j < \alpha_{j+1}^{(i)}$ for every $j$ and $1 \leqslant i \leqslant q-1$. Denote $\bar{\xi}_j = \beta_j - \alpha_j^{(1)}$ and $\bar{\eta}_j = \alpha_{j+1}^{(1)} - \beta_j$, $(j \geqslant 0)$.

Next we redefine $X_n$, i.e., by the help of $Z_n$ we determine a random walk $X_n$ which has the prescribed transition probabilities $P$. For the simplicity of exposition we suppose $X_0 = b_1$. In other cases the proof can be easily modified.

**Definition: $\gamma_j, \delta_j, \Theta_j, \xi_j, \eta_j$.** For the random walk $X_n$ to be defined we denote by $\gamma_j$ and $\delta_j$ ($\gamma \geqslant 0$) the time of the $j$th entrance into the set $K_M$ and $\mathbb{Z}^d \backslash K_M$, respectively ($\gamma_0 = 0$). Set, moreover, $\Theta_j = X_{\gamma_j}(\Theta_0 = b_1), \xi_j = \delta_j - \gamma_j$, $\eta_j = \gamma_{j+1} - \delta_j$.

**Definition: $X_n$.** In the time intervals $[\gamma_j, \delta_j)$ the definition of $X_n$ is very simple: since $X_{\gamma_j} = b_i$ for some $i \in \{1, \ldots, q\}$ we just set $X_n = Z_{\alpha_j^{(i)} + n - \gamma_j}$ if $\gamma_j \leqslant n < \delta_j$ and $\delta_j$ is of course $\gamma_j + \alpha_j^{(i+1)} - \alpha_j^{(i)}$ if $1 \leqslant i \leqslant q - 1$ and equals $\gamma_j + \beta_{j+1} - \alpha_j^{(q)}$ if $i = q$. By definition $X_{\delta_j} \in \partial K^{(i)}$. By a unique Euclidian transformation $\varphi$ of $\mathbb{Z}^d$ we can reach that $\varphi(Me_1) = X_{\delta_j} - 1$ and $\varphi(K_M) \supset K_N$. Then $X_n = \varphi(Z_{\beta_j + n - \delta_j})$ if $\delta_j \leqslant n < \delta_j + \alpha_{j+1}^{(1)} - \beta_j$. By definition $Z_{\alpha_{j+1}^{(1)} - 1} \in \partial(\mathbb{Z}^d \backslash K_M)$ and we can now take a simple, symmetric random walk $V_j(0), V_j(1), \ldots$ starting from the point $Z_{\alpha_{j+1}^{(1)} - 1}$ and independent of $Z_n$ and of $X_0, X_1, \ldots, X_{\delta_j}$. We can take this random walk until the first time $\kappa_j$ when $\varphi(V_j(\kappa_j)) \in \partial K_N$. Then we put $\gamma_{j+1} = \delta_j + \alpha_{j+1}^{(1)} - \beta_j + \kappa_j$ and we define $X_n = \varphi(V_j(n - \delta_j - \alpha_{j+1}^{(1)} + \beta_j))$ whenever $\delta_j + \alpha_{j+1}^{(1)} - \beta_j \leqslant n \leqslant K_j$. Now clearly $X_{\gamma_{j+1}} = b_i$ for some $i$ and the definition of $X_n$ can be continued.

It is easy to see that the distribution of the process defined above coincides with the distribution of the original process and also that $Z_n$ satisfies all the properties stated before (∗). ■

## NOTE ADDED IN PROOF

After this paper had been submitted, there appeared a work by J. M. Harrison and L. A. Shapp which investigated in detail the example of Section 2 (On Skew Brownian Motion, *Annals of Probability* **2**:309–313 (1981)).

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